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Empirical Bayes Estimation for Markov Chain Models of Drought Events in Peninsular Malaysia

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Abstract. This study employs empirical Bayes method to estimate the transition probability matrix of Markov chain. The transition probability is used to determine drought characteristics for 35 rainfall stations in Peninsular Malaysia. The result reveals that non-drought condition is more persistent than the other drought conditions. The result also shows that the middle area of Peninsular Malaysia experiences longer non-drought condition with higher probability compared to other regions. Meanwhile, western area experiences moderate drought condition, more frequent with shorter duration.

Keywords: Drought categories, empirical Bayes, Markov chain, the transition probability matrix.
PACS: 02.50.-r.

INTRODUCTION

Estimation of the transition probability is important in Markov chain modeling. Many researchers employed the maximum likelihood method to estimate the transition probability [1-7]. Another method that could also be used is the empirical Bayes method, as proposed by Meshkani & Billard [8]. There have been cases which the maximum likelihood method results in zero transition probability estimator but this is not true with empirical Bayes method.

Markov chain models have been used by Banik et al. [9] in determining drought-proneness for weekly rainfall data in India. Paulo et al. [5] and Paulo and Pereira [6-7] applied Markov chain model to predict drought class transition. Mishra et al. [4] also used Markov chain model to investigate the probability of drought duration and persistence. Meantime, Deni et al. [10] determined the optimum order of the Markov chain model for daily rainfall occurrences in Peninsular Malaysia. In this study, empirical Bayes method will be employed to describe drought characteristics in Peninsular Malaysia. The drought characteristics considered are a drought persistence, probability of drought, drought duration and mean recurrence time of drought.

DATA AND METHODS

This study method, the application of the monthly rainfall amount data (in mm) from 35 rainfall stations in Peninsular Malaysia for the period of 1970 – 2008 (TABLE (1)), obtained from the Drainage and Irrigation Department and Malaysian Meteorological Department. Three stations have complete data while each of the other 32 stations have less than 10% of missing data. The missing data were replaced by estimation using the normal ratio and modified normal ratio methods [11-12]. Based on Gustafson-Kessel fuzzy clustering method and adjustment of region [13-14], thirty five rainfall stations could be grouped into six regions (FIGURE 1). The drought state is classified into four states based on the standardized precipitation index (SPI) [15] as shown in TABLE (2). SPI values are calculated by using the approximation as provided by Abramowitz and Stegun [16] as follows:

Firstly, fitting the Gamma probability density function of the frequency distribution of monthly rainfall data for each study station. The cumulative probability function $F(x)$ for the rainfall amount (x) formed from the Gamma distribution. Since the gamma distribution is not defined for $x = 0$ and rainfall value may contain zero, the cumulative probability function of X becomes $H(x) = p + (1 - p)F(x)$, where $p = Prob(x = 0) > 0$. Finally, SPI values are calculated as follows

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$$\text{SPI} = -\left(w - \frac{c_0 + c_1 w + c_2 w^2}{1 + d_1 w + d_2 w^2 + d_3 w^3}\right),\ \text{for } 0 < H(x) \leq 0.5 \tag{1}$$

$$\text{SPI} = -\left(w - \frac{c_0 + c_1 w + c_2 w^2}{1 + d_1 w + d_2 w^2 + d_3 w^3}\right),\ \text{for } 0 < H(x) \leq 0.5 \tag{2}$$

where $w = \sqrt{ln\left[\frac{1}{(H(x))^2}\right]}$, for $0 < H(x) \leq 0.5$ and $w = \sqrt{ln\left[\frac{1}{(1-H(x))^2}\right]}$, for $0.5 < H(x) < 1$,

$c_0 = 2.515517$, $c_1 = 0.802853$, $c_2 = 0.010328$, $d_1 = 1.432788$, $d_2 = 0.189269$, $d_3 = 0.001308$.

Further, each station in the period considered, data of drought categories for the current month and the following month are arranged into transition count matrix and categorized as non-drought, near-normal, moderate and severe. Data that are used in this study are also found to satisfy homogeneity property and follows the first order Markov chain. Each the homogeneity property and the first order Markov chain of data verified using the likelihood ratio test.

TABLE (1). Name and location of rainfall stations in Peninsular Malaysia.

Code	Name of Rainfall Stations	Latitude (N)	Longitude (E)	Code	Name of Rainfall Stations	Latitude (N)	Longitude (E)
1	Alor Setar	06 06 20	100 23 30	19	Ldg. Johol	02 36 10	102 19 10
2	Baling	05 35 00	100 44 10	20	Kg. Sawah Lebar	02 45 20	102 15 50
3	Bkt. Bendera	05 25 25	100 16 15	21	Petaling Kuala Klawang	02 56 40	102 03 55
4	Jeniang Klinik	05 48 50	100 37 55	22	Ldg. New Rompin	02 43 10	102 30 45
5	Sg. Pinang	05 23 30	100 12 45	23	Sikamat Seremban	02 44 15	101 57 20
6	Dabong	05 22 40	102 00 55	24	Semenyih	02 53 55	101 52 13
7	Gua Musang	04 52 45	101 58 10	25	Sg. Manggis	02 49 35	101 32 30
8	Kg. Aring	04 56 15	102 21 10	26	Endau	02 39 00	103 37 15
9	Kampar	04 18 20	101 09 20	27	Chin-Chin	02 17 20	102 29 30
10	Ldg. Bikam	04 02 55	101 18 00	28	Johor Bahru	01 28 15	103 45 10
11	Sg. Bernam	03 41 53	101 20 60	29	Kota Tinggi	01 45 50	103 43 10
12	Sitiawan	04 13 05	100 42 00	30	Ldg. Sg. Labis	02 23 05	103 01 00
13	Telok Intan	04 01 00	101 02 10	31	Dungun	04 45 45	103 25 10
14	Ampang	03 09 20	101 45 00	32	Kemaman	04 13 55	103 25 20
15	Ldg. Edinburgh	03 11 00	101 38 00	33	Kg. Dura	05 04 00	102 56 30
16	Genting Klang	03 14 10	101 45 10	34	Paya Kangsar	03 54 15	102 26 00
17	Gombak	03 16 05	101 43 45	35	Pekan	03 33 40	103 21 25
18	Kg. Sg. Tua	03 16 20	101 41 10				

TABLE 2. Classification of drought based on SPI

SPI Values	Drought Categories
SPI ≥ 0	Non-drought
$-1 < \text{SPI} < 0$	Near normal
$-1.5 < \text{SPI} \leq -1$	Moderate
SPI ≤ -1.5	Severe

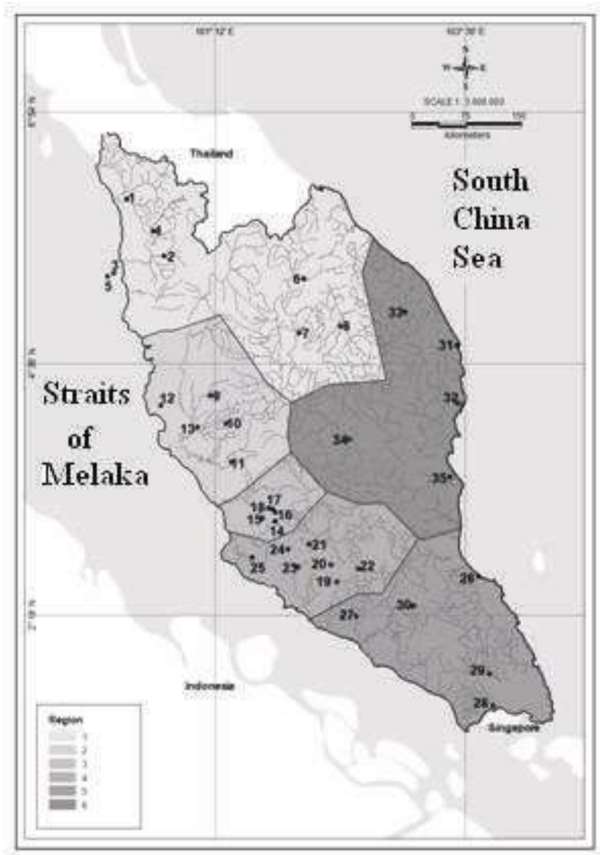

FIGURE 1. Location of rainfall stations used in this study

Empirical Bayes estimator for Transition Probability Matrix of Markov Chain

A Markov chain $\{X_t,\ t = 0, 1, 2, \cdots\}$ is a stochastic process with property that the process value at time $t + 1$, denoted X_{t+1}, depends only on its value at time t or X_t, such that for every t and all states $i_0, \ldots, i_{t+1}$, we have

$$Prob(X_{t+1} = i_{t+1} | X_t = i_t, \cdots, X_0 = i_0) = Prob(X_{t+1} = i_{t+1} | X_t = i_t). \tag{3}$$

Let $Prob(X_{t+1} = j | X_t = i) = p_{ij}$ be the transition probability from state i at time t to state j at time $t + 1$, then p_{ij} could be represented in the transition probability matrix form, $\boldsymbol{P}$, as follows

$$\boldsymbol{P} = [p_{ij}] = \begin{bmatrix} p_{11} & \cdots & p_{1s} \\ \vdots & \ddots & \vdots \\ p_{s1} & \cdots & p_{ss} \end{bmatrix}, i, j = 1, \ldots, s,\ 0 \le p_{ij} \le 1,\ \sum_{j=1}^{s} p_{ij} = 1, i = 1, \ldots, s \text{ and } s \text{ is the number of states.}$$

Empirical Bayes Estimation

Let $F = [f_{ij}]$ represents the transition count matrix. The row vector of F is denoted by $F_i = [f_{i1} \ldots f_{is}]$ which is assumed follow the multinomial distribution with parameters $P_i = [p_{i1}, \ldots, p_{is}]$. Meshkani & Billard [8] assumed the matrix beta distribution, $h(P|\beta)$, as a conjugate prior distribution for P and a matrix $\beta = [\beta_{ij}]$ is the matrix of parameters. The prior predictive distribution of F is

$$L(F) = \prod_{i=1}^{s}\left[\prod_{j=1}^{s}\frac{(F_{i.})!}{f_{ij}!}\right]\left(\prod_{i=1}^{s}\left[\frac{\Gamma(\beta_{i.})}{\Gamma(\beta_{i.}+F_{i.})}\prod_{j=1}^{s}\left\{\frac{\Gamma(\beta_{ij}+f_{ij})}{\Gamma(\beta_{ij})}\right\}\right]\right), \tag{4}$$

and the posterior distribution of P for a given F, is $g(P|F) \propto \prod_{i,j}(p_{ij})^{\beta_{ij}+f_{ij}-1}$, which $g(P|F)$ follows the matrix beta distribution with a parameter $\beta^* = [\beta_{ij} + f_{ij}]$. Bayes estimator for P is the posterior mean of P for a given F, that is $P_{\mathrm{B}} = \frac{(\beta_{ij}+f_{ij})}{(\beta_{i.}+F_{i.})}$, where $F_{i.} = \frac{f_{ij}}{\sum_{j=1}^{s} f_{ij}}$. In empirical Bayes method, we need to estimate a prior parameter for β_{ij} by using the observed data. Meshkani and Billard [8,17-18] proposed the estimator for β_{ij} based on the past data. Suppose there are m rainfall stations studied which each station has the transition probability matrix $\boldsymbol{F} = [f_{ij;k}]$, $k = 1, ..., m$ and $i, j = 1, ..., s$, if $f_{ij;k}$ as the current data, then the past data refers to the data set $\{f_{ij;l}\}$, $l = 1, ..., m$, and $l \neq k$. Based on the current data and information from the past data, we could obtain estimator for the transition probability matrix. Meshkani and Billard [8] used method of moments to estimate a parameter β_{ij}, as follows:

Let $M_{ij} = \frac{f_{ij}}{F_{i.}}$ and $Y_{ij} = M_{ij}(1 - M_{ij})$, $F_{i.} \neq 0$, then $\mathrm{E}[M_{ij}] = \frac{\beta_{ij}}{\beta_{i.}}$ and $\mathrm{E}[Y_{ij}] = \mathrm{E}[M_{ij}]\{1 - \mathrm{E}[M_{ij}]\}\left(\frac{\beta_{i.}}{\beta_{i.}+1}\right)$. The moments estimator for β_{ij} could be determined by using the sample average of the past data. Thus, estimator for $\beta_{ij;k}$ is

$$b_{ij;k} = \hat{\beta}_{ij;k} = \frac{\bar{M}_{ij;k}\bar{Y}_{ij;k}}{\{[\bar{M}_{ij;k}(1-\bar{M}_{ij;k})] - \bar{Y}_{ij;k}\}} \tag{5}$$

and $b_{i.;k} = \hat{\beta}_{i.;k} = \sum_{j=1}^{s} b_{ij;k}$. Therefore, we obtain empirical Bayes estimator of the transition probability matrix for each rainfall station in the corresponding region, as follows:

$$\boldsymbol{P}_{\mathrm{EB;M}} = [\hat{p}_{ij;k}] = \frac{(f_{ij;k}+b_{ij;k})}{(F_{i.;k}+b_{i.;k})} \tag{6}$$

The Mean Residence Time

Let p_{jj} denote the transition probability of Markov chain $\{X_t\}$ with drought category j and R_j is the residence time for any category j and for n is a number of months

$$\begin{aligned} Prob(R_j = n) &= Prob(X_{t+1} = j|X_t = j) \ldots Prob(X_{t+n-1} = j|X_{t+n-2} = j)\, Prob(X_{t+n} \neq j|X_{t+n-1} = j) \\ &= (p_{jj})^{(n-1)}(1 - p_{jj}). \end{aligned}$$

(7)

Meanwhile, the mean residence time for any drought category j is given by $\mathrm{E}(R_j|X_t) = \frac{1}{(1-p_{jj})}$.

The Mean Recurrence Time

The first passage time from i to j denotes the time taken for a process to move for the first time in drought category i to category j. The mean first passage time from state i to state j, M_{ij}, is defined as

$$M_{ij} = 1 + \sum_{\substack{k=1 \\ k\neq j}}^{s} p_{ik}\, M_{kj}, \quad \text{for every} \quad i, j = 1, \ldots, s. \tag{8}$$

In the matrix form, Eq. (8) becomes $\boldsymbol{M} = \boldsymbol{E} + \boldsymbol{P}(\boldsymbol{M} - \boldsymbol{M}_{\mathrm{d}})$, where $\boldsymbol{M}$ is a matrix with elements M_{ij}, $\boldsymbol{E}$ is a unit matrix, $\boldsymbol{P} = [p_{ij}]$, and $\boldsymbol{M}_{\mathrm{d}}$ is the diagonal matrix whose elements, $(\boldsymbol{M}_{\mathbf{d}})_{jj} = M_{jj}$. The mean first passage time, M_{jj}, is called the mean recurrence time for any drought category j, that is, the average time required to leave the initial category j before returning to the same category.

TABLE 3. The Transition Probability Matrices of Drought Categories for Each Station

Code	State	ND	NN	M	S	Code	State	ND	NN	M	S	Code	State	ND	NN	M	S
1	ND	0.524	0.379	0.066	0.031	13	ND	0.396	0.390	0.137	0.078	25	ND	0.598	0.303	0.071	0.028
	NN	0.740	0.152	0.079	0.029		NN	0.724	0.190	0.048	0.038		NN	0.450	0.255	0.249	0.045
	M	0.393	0.411	0.032	0.175		M	0.263	0.303	0.114	0.221		M	0.430	0.271	0.103	0.195
	S	0.423	0.341	0.089	0.146		S	0.396	0.400	0.092	0.112		S	0.294	0.381	0.068	0.256
2	ND	0.600	0.298	0.062	0.041	14	ND	0.481	0.329	0.067	0.123	26	ND	0.489	0.366	0.070	0.076
	NN	0.773	0.157	0.040	0.028		NN	0.642	0.231	0.061	0.066		NN	0.545	0.246	0.118	0.090
	M	0.310	0.404	0.134	0.152		M	0.429	0.210	0.128	0.133		M	0.641	0.211	0.073	0.076
	S	0.242	0.309	0.256	0.194		S	0.389	0.302	0.167	0.142		S	0.502	0.394	0.032	0.072
3	ND	0.435	0.457	0.032	0.042	15	ND	0.154	0.327	0.009	0.010	27	ND	0.391	0.365	0.025	0.110
	NN	0.778	0.147	0.037	0.038		NN	0.787	0.167	0.019	0.026		NN	0.559	0.228	0.077	0.136
	M	0.393	0.383	0.066	0.159		M	0.291	0.307	0.167	0.135		M	0.657	0.224	0.062	0.058
	S	0.182	0.624	0.057	0.077		S	0.343	0.320	0.230	0.108		S	0.501	0.320	0.074	0.065
4	ND	0.570	0.322	0.070	0.037	16	ND	0.492	0.359	0.070	0.079	28	ND	0.556	0.322	0.069	0.053
	NN	0.757	0.169	0.043	0.030		NN	0.477	0.347	0.133	0.042		NN	0.587	0.234	0.083	0.096
	M	0.343	0.296	0.179	0.183		M	0.482	0.219	0.162	0.137		M	0.499	0.322	0.089	0.090
	S	0.256	0.326	0.113	0.305		S	0.588	0.275	0.108	0.036		S	0.498	0.421	0.030	0.052
5	ND	0.558	0.331	0.071	0.040	17	ND	0.497	0.370	0.067	0.064	29	ND	0.513	0.306	0.115	0.067
	NN	0.785	0.147	0.040	0.030		NN	0.635	0.243	0.038	0.116		NN	0.428	0.507	0.032	0.033
	M	0.377	0.362	0.083	0.178		M	0.231	0.084	0.053	0.612		M	0.560	0.254	0.100	0.086
	S	0.274	0.341	0.116	0.268		S	0.347	0.331	0.209	0.122		S	0.586	0.292	0.047	0.075
6	ND	0.548	0.330	0.077	0.045	18	ND	0.539	0.306	0.056	0.099	30	ND	0.493	0.348	0.064	0.095
	NN	0.776	0.150	0.039	0.035		NN	0.608	0.259	0.059	0.074		NN	0.576	0.229	0.104	0.090
	M	0.317	0.402	0.067	0.214		M	0.467	0.232	0.147	0.154		M	0.558	0.300	0.076	0.066
	S	0.443	0.336	0.079	0.141		S	0.171	0.691	0.091	0.047		S	0.554	0.316	0.098	0.032
7	ND	0.530	0.335	0.069	0.058	19	ND	0.643	0.275	0.053	0.029	31	ND	0.627	0.261	0.047	0.065
	NN	0.979	0.015	0.003	0.003		NN	0.500	0.273	0.172	0.054		NN	0.487	0.320	0.063	0.130
	M	0.417	0.369	0.043	0.171		M	0.465	0.278	0.065	0.193		M	0.402	0.471	0.046	0.081
	S	0.361	0.372	0.085	0.182		S	0.356	0.312	0.174	0.158		S	0.365	0.228	0.092	0.315
8	ND	0.549	0.338	0.078	0.039	20	ND	0.590	0.281	0.093	0.036	32	ND	0.540	0.315	0.100	0.034
	NN	0.779	0.143	0.042	0.036		NN	0.428	0.225	0.249	0.078		NN	0.511	0.209	0.152	0.127
	M	0.209	0.642	0.029	0.120		M	0.377	0.412	0.055	0.156		M	0.238	0.721	0.024	0.018
	S	0.316	0.363	0.099	0.222		S	0.538	0.353	0.086	0.123		S	0.264	0.238	0.092	0.406
9	ND	0.452	0.200	0.277	0.071	21	ND	0.582	0.298	0.085	0.030	33	ND	0.528	0.272	0.053	0.046
	NN	0.837	0.109	0.029	0.026		NN	0.551	0.231	0.160	0.058		NN	0.536	0.294	0.075	0.095
	M	0.461	0.298	0.073	0.167		M	0.438	0.267	0.091	0.215		M	0.451	0.473	0.042	0.034
	S	0.399	0.394	0.086	0.121		S	0.328	0.302	0.071	0.298		S	0.290	0.237	0.109	0.394
10	ND	0.642	0.158	0.143	0.056	22	ND	0.588	0.287	0.090	0.034	34	ND	0.764	0.188	0.027	0.024
	NN	0.827	0.132	0.037	0.017		NN	0.501	0.256	0.179	0.064		NN	0.544	0.261	0.093	0.102
	M	0.462	0.288	0.110	0.141		M	0.409	0.368	0.070	0.154		M	0.410	0.500	0.044	0.036
	S	0.401	0.399	0.063	0.137		S	0.434	0.336	0.083	0.147		S	0.236	0.139	0.084	0.391
11	ND	0.500	0.199	0.143	0.158	23	ND	0.580	0.312	0.067	0.041	35	ND	0.561	0.339	0.055	0.045
	NN	0.696	0.166	0.069	0.069		NN	0.505	0.302	0.150	0.043		NN	0.574	0.279	0.073	0.076
	M	0.438	0.318	0.038	0.156		M	0.397	0.283	0.154	0.162		M	0.445	0.457	0.071	0.037
	S	0.534	0.379	0.060	0.076		S	0.577	0.206	0.081	0.142		S	0.286	0.302	0.083	0.326
12	ND	0.541	0.225	0.150	0.084	24	ND	0.604	0.298	0.063	0.035						
	NN	0.664	0.235	0.050	0.047		NN	0.524	0.336	0.190	0.050						
	M	0.491	0.255	0.039	0.166		M	0.405	0.317	0.045	0.232						
	S	0.216	0.644	0.045	0.095		S	0.335	0.201	0.092	0.165						

RESULTS AND DISCUSSIONS

The empirical Bayes estimators of the transition probability for each station are given in TABLE (3). This table gives the probability of drought persistence for each station with different drought states, which was obtained from the diagonal of each transition probability matrix. The drought persistence value represents the probability of the same drought category in two consecutive months. TABLE (3) also shows that, in generally, persistence of non-drought state has a highest transition probability than other states, where its probability is greater than 0.5. Therefore, near-normal state has the second highest persistence probability.

Based on spatial distribution of non-drought probability, FIGURE 2 (a) shows that the middle area of Peninsular Malaysia has non-drought probability higher than other areas, with probability greater than 0.6. This area also experienced longer non-drought duration, which was about 3 – 4 months with mean recurrence time is about 1.5 months (FIGURE 3 (a) and FIGURE 4 (a)). The longer duration for near normal condition occurred in southern area for about two months with probability between 0.3 and 0.4 and mean recurrence time of about 2 – 3 months (FIGURE 2 (b), FIGURE 3 (b) and FIGURE 4 (b)). In the northern and western areas, mean recurrence time of moderate drought is between 0.5 and 1 years with the duration is approximately 1.1 months (FIGURE 3 (c) and FIGURE 4 (c)). Meanwhile, the longer severe drought duration occurred in middle area, that is about 2 months, but its probability is smaller, and this condition also has mean recurrence time about 8 – 12 months (FIGURE 2 (d), FIGURE 3 (d) and FIGURE 4 (d)).

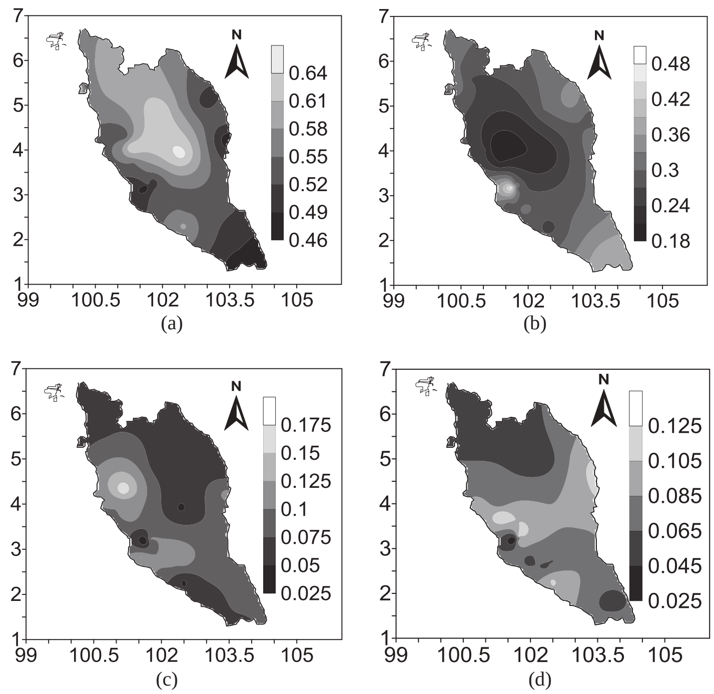

FIGURE 2. Spatial distribution of drought probability: (a) Non-drought; (b) Near-normal: (c) Moderate drought; and (d) Severe drought.

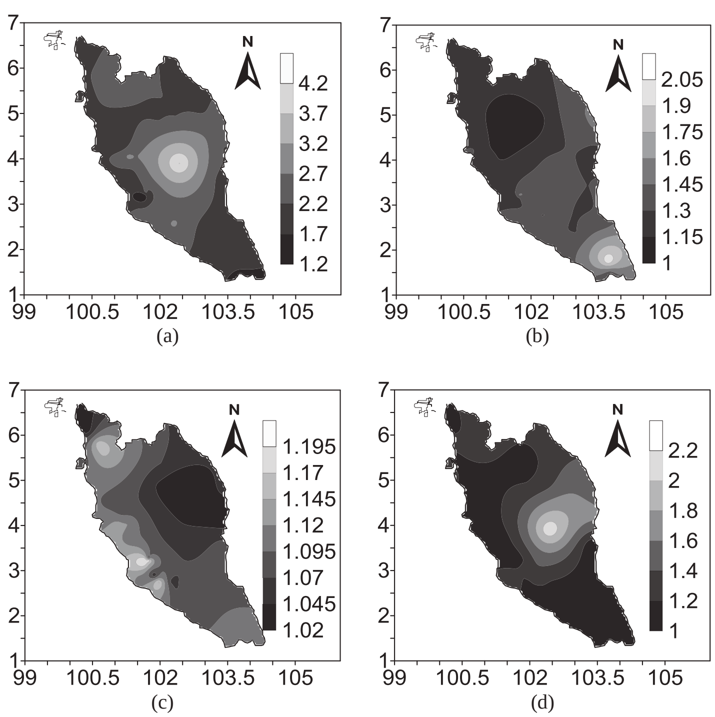

FIGURE 3. Spatial distribution of the mean residence time of drought (months): (a) Non-drought; (b) Near-normal: (c) Moderate drought; and (d) Severe drought.

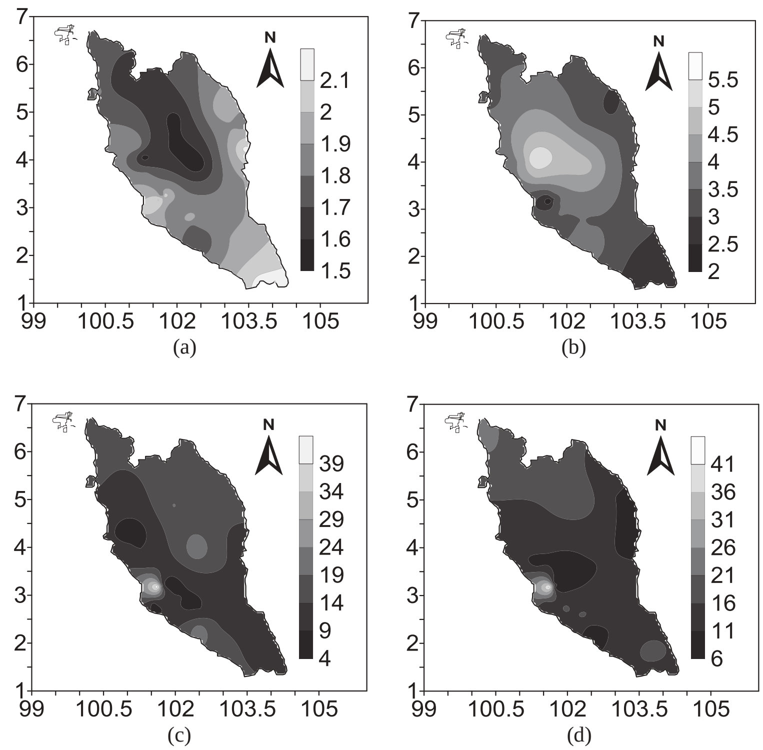

FIGURE 4. Spatial distribution of the mean recurrence time of drought (months): (a) Non-drought; (b) Near-normal: (c) Moderate drought; and (d) Severe drought.

CONCLUSION

In this study, the transition probability matrix estimator of Markov chain has been determined using empirical Bayes method. Based on the transition probability matrix obtained, the non-drought event was more persistent than other drought categories. Spatial distribution of drought characteristics revealed that middle area of peninsular Malaysia experienced longer non-drought event with probability greater than other areas, while for severe drought event, mean recurrence time of this area is smaller than one year. The southern area experienced near normal condition, with duration about two months and its probability is approximately 0.3. The northern and western areas has mean recurrence time of moderate drought ranging from 0.5 to 1 year with duration of about one month.

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