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To cite this article: I Minggi *et al* 2021 *J. Phys.: Conf. Ser.* **2123** 012046

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Language Constraints in Constructing Arguments for Mathematical Proofs

I Minggu¹, Alimuddin¹, and Sabri^{1*}

¹ Department of Mathematics, Universitas Negeri Makassar, Jl. Daeng Tata, Makassar 90224, Indonesia

*Email: sabri@unm.ac.id

Abstract. A learning trajectory for constructing mathematical proof has been developed. The trajectory is to provide the students with a step-by-step procedure in constructing arguments for proving mathematical statements. However, in proving activities, the students were found to encounter difficulties in completing a deductive axiomatic argument constituting an accepted mathematical proof. An investigation has been conducted to explore the problems the students experienced in constructing proofs. It was found that they faced language constraints in constructing mathematical arguments. They encountered challenges in how to correctly express the mathematical statements in their constructed proofs.

Keywords: Deductive arguments, mathematical proof, language constraints.

1. Introduction

Proving mathematical statements is an integral part of learning mathematics. The importance of proof and proving in the study of mathematics has been realized since ancient times [1]. Despite the important position of proof in mathematics, in particular university mathematical study, research [e.g. 2, 3-8] found that undergraduate students' understanding of proof is problematic and they encountered difficulties in developing proofs. There is the inadequacy of research focusing on the students in terms of how they start to construct proof and the effect of particular teaching and learning approaches on the students' understanding. Regarding the phenomenon of proving difficulties, mathematics education experts claim to know where the students and the mathematicians are, respectively, but they do not know how to improve the university students' capability of constructing mathematical proofs [9].

Hitherto, we have formulated and experimented with a learning trajectory of proving for students of the mathematics department [10, 11]. The learning trajectory is to provide the students with a step-by-step procedure in constructing arguments for proving mathematical statements. However, in proving activities, some students were found to encounter difficulties in completing a deductive axiomatic argument constituting an accepted mathematical proof. Many studies have investigated the activities of proving and they identified some main difficulties experienced by the students, for instance, the nescience of how to start an argument of proof, the lack of understanding and using concept definitions, the incapability of using logical rules and quantifiers, or the discrepancy between the colloquial language and the mathematical language to name a few [2, 6, 12, 13]. In this research, the focus is on the language difficulties faced by the students. It aims at investigating the language constraints experienced by the students in their attempt to construct a mathematical argument of proof.



2. Theoretical review

Scholars put a strong emphasis on the central position of mathematical proof in mathematics learning [14-17]. Proving is the essence of mathematical activities, and therefore, it plays a central role in the mathematical study [18]. Mathematical proof has characteristics distinguishing mathematics from other scientific disciplines or other human endeavors [19, 20]. In mathematics, the proof is a deductive argument justifying the truth of mathematical statements derived from other true statements and also verifying the mathematical structures within the statements [21]. In proving activities, students show their ability to develop logical arguments, to distinguish examples from non-examples supporting their arguments, their weaknesses in reasoning, and their misconceptions. Epp [22, 23] claims that one approach appropriate to develop students' mathematical thinking is involving them in meaningful activities of constructing or verifying mathematical proofs. In a similar vein, Hanna [24] suggests that mathematical proof is effective in developing mathematical conceptions. Mathematical learning without proving is not reflecting the real mathematical theories and practices [24].

The learning trajectory for proving proved to function for students [11]. Following the steps, starting from understanding the statement to prove, choosing the proving strategies to employ, constructing the argument, and then verifying the argument validity, leads the students to produce an argument constituting an accepted proof. In the process, it was found that the lecturer should be there to facilitate the students' activity by providing them with proportional assistance or 'scaffolding' [25]. Arguments constructed sometimes still need some refinement to satisfy the qualities, such as the clarity and fluency [26], of accepted proof by mathematicians. On the other hand, some students struggled in devising their proof arguments. Some of them seemed not to know how to start an argument, what to write first, or not know how to express their mathematical ideas. To a certain extent, it is all related to language.

Some studies claim that the problems experienced by students in proving could be related to the sudden introduction of proof in high school mathematics [27]. Many students in the course units which are proof-oriented, such as, geometry [28], introduction to proof [6], real analysis [29], and abstract algebra [8], still could not construct simple or routine proofs. Further, students generally do not know what should be expressed in a proof [30] and they could not verify whether a proof is valid or not [31]. In particular, many students attempt to justify the truth of mathematical propositions by using examples or showing cases even after the clear instruction regarding the general property of proofs in mathematics [16]. It is also found that the students who understand some of the roles the mathematical proof plays are less likely to take empirical arguments as proof compared to their counterparts who do not [32]. Besides the aforementioned difficulties in proving, the extant literature also identifies the other main difficulties in terms of the students' conception of proof and the need for proving [33] or the students' understanding of proving methods [34]. Some difficulties mentioned earlier have clear a relationship to language matters.

Linguistic knowledge lays the foundation for mathematical knowledge [35]. We start to learn mathematics in language, we either proceed smoothly or struggle along the way all because of language, and we achieve the learning outcomes which are assessed in the language [36]. The relationship of language to mathematics is undeniable [37]. A language is a mediating tool between mathematical concepts and principles and for the development of mathematical systems [38]. The dynamic of symbol manipulations, equation solutions, and relationships among mathematical objects constitute problems for the learners [39]. Although mathematics is commonly considered universal, independent from the culture [40, 41], and employs various notations generally understood across cultures [37], the language of mathematics could confuse the students who take into account their previous knowledge [38]. language is an internal constraint in mathematics learning [42]. In conceptual mathematics unit courses, the focus on proofs and arguments emphasizing ideas expressed correctly, precisely, clearly, and concisely places the grammar, logic, and rhetoric in a central position [39].

Constructing an argument for proving the truth of mathematical statements involves several features. In this research, we use the notion of argument for mathematical proof defined by Lew and Mejia Ramos [43] as a genre entailing the characteristic of formality, the structure of language, and the

communicative objective. The vernacular is different from mathematical language; therefore, understanding mathematical proofs requires the mastery of the language of the discipline [44]. Here, we consider mathematical language as a synonym to the mathematics register defined by Halliday [45] which contains a particular language function, along with technical vocabularies, phrases, signs, and structures. The qualities that should be considered in mathematical language range from the avoidance of passive voice [46], the use of appropriate grammar, correct diction, and succinct expression [47, 48], the use of mathematical symbols grammatically [46, 47], to the use of clear referents [49].

Previous research in undergraduate mathematics education has found the students to encounter difficulties in understanding reading [50], developing [8], justifying [31] proof arguments. As mentioned earlier, the difficulties in proving which cover the incapability of starting to write proof and using rules of logic, the lack of concept understanding, and the inability to distinguish the mathematical language from the vernacular are all about language. Moore [6] emphasizes the unfamiliarity with symbolization and language of mathematics as one of the main sources of students' problems.

The lack of language knowledge incurs constraints or barriers for students in their proving activities. The problem with language use in mathematics, to some extent, is due to the different way of thinking and mental capacity demanded in mathematics compared to that in ordinary language [51]. Further, precision is central in mathematical language, while this quality is sacrificed in ordinary language for the sake of utility. There are three linguistic features of mathematics distinguishing it from the ordinary use of language [52]. The first is the semiotic systems that combine the symbolic and visual representations bearing meaning different from that in oral or written language. The second is the dense noun phrases, technical vocabularies, and grammar structures used in mathematical relationships. And the last is the use of conjunctions whose meanings are different from that in daily language and the implicit logical assumptions linking elements in mathematical relationships. These all form the analytical framework for this research.

3. Methods

This study is qualitative interpretive research. It is designed as a modified teaching experiment [53]. It was implemented in several classes of proof-oriented mathematics courses, namely, Calculus, the Fundamentals of Mathematics, and Real Analysis, at one state university located in the province of Sulawesi Selatan, Indonesia. The participants were students of the mathematics department from various years of study.

In the lecture sessions, students were given problems of proving including constructing an argument for proving the truth of mathematical statements (propositions, theorem, lemmas, or corollaries) or justifying the validity of proof arguments. Some students' works were given in the written form and some others were given in recorded video of them working aloud on the given problems. The data were analyzed by the content analysis method aiming at providing insights into the phenomena under investigation [54, 55]. The analysis was implemented by examining text-based data to develop a description or interpretation of the texts [56].

4. Findings and discussion

We present the results of the data analysis regarding the language constraints the students experienced in proving activities. These constraints are presented based upon the three linguistic characteristics of mathematics that distance it from the ordinary language [52]. The discussion section will follow the presentation of the findings.

The data analysis reveals the challenges the students experienced concerning the language of mathematics in proving activities. With regard to the system of meaning-making where the symbolic representations and visual images are used with the specific mathematical meaning, the students encountered the problem of how to represent parts of their arguments in correct symbolic representations or visual images. The statements they include in their arguments which they intended

to constitute a proof contain incorrect symbols. Proving the non-existence of the limit of a function, namely, $\lim_{x \rightarrow 0} \sin \frac{1}{x}$, one student has the following proof.

“Limit x menuju nol sin seper x itu tidak ada. Kenapa? Karena ada satu nilai sin seper x_n itu konvergen ke satu sedangkan untuk y_n itu konvergen ke nol, walaupun nilai x_n dan nilai y_n itu sama, sama-sama konvergen ke nol, tetapi untuk nilai sin seper x_n dan sin seper y_n , yang satu menuju konvergen satu dan y_n menuju ke nol, sehingga kita dapat menyimpulkan bahwa untuk limit x menuju 0 sin seper x itu tidak ada.”

“Limit x approaching zero of sine one over x does not exist. Why? Because there is one value of that sine one over x convergent to one, whereas for y_n that is convergent to zero, although those values of x_n and y_n are equal, both are convergent to zero, but for the values of sine one over x_n and sine one over y_n , one is toward convergent to equal to one and y_n is toward zero, so we could conclude that for limit x approaching zero of sine one over x it does not exist.”

$x_n = \frac{2}{(4n-3)\pi}$ konvergen, 0, dan
 $\sin \frac{1}{x_n} = \sin \left(\frac{(4n-3)\pi}{2} \right) = 1$
 $x_n = \frac{(4n-3)\pi}{2}$
 $y_n \rightarrow 0$
 $\frac{1}{y_1} = \pi, \sin \left(\frac{1}{y_1} \right) = \sin \pi = 0$
 $y_1 = \frac{1}{\pi}$
 $\frac{1}{y_2} = 2\pi, \sin \left(\frac{1}{y_2} \right) = \sin 2\pi = 0$
 $y_2 = \frac{1}{2\pi}$
 $y_n = \left(\frac{1}{n\pi} \right)$ konvergen 0 + lagi, $\sin \left(\frac{1}{y_n} \right)$
 $= \sin n\pi = 0$
 Ada dua barisan konvergen (x_n) = 2 konvergen 0
 dan $y_n = \left(\frac{1}{n\pi} \right)$ konvergen 0
 Tetapi, $\sin \left(\frac{1}{x_n} \right)$ konvergen ke 1 dan $\sin \left(\frac{1}{y_n} \right) \rightarrow 0$

Figure 1. Andy’s proof

In this proof, it could be observed that Andy did not write some of the symbolic expressions in accordance with the conventional rules in expressing sequences and series and their convergence. At the very beginning of the proof, he wrote $x_n = \frac{2}{(4n-3)\pi} \xrightarrow{\text{convergent}} 0$ to state that the sequence (x_n) is convergent to 0. The convention for expressing a sequence (x_n) convergent to some real number L introduced in the lecture or in the textbook is $(x_n) \rightarrow L$, $\lim(x_n) = L$, or $\lim X = L$, where X , (x_n) , or $(x_n : x \in \mathbb{N})$ is the symbol of a sequence. These symbolic representations of a convergent sequence contain an implicit assumption that basically it is a limit, a limit at infinity. In certain textbooks, however, it is said that we could sometimes use $x_n \rightarrow L$ as an *intuitive indication* that the values of the sequence “ x_n ” approach L when $n \rightarrow \infty$ [57]. In short, the word “convergent” is not necessary for the symbolization. The arrow “ $\rightarrow$ ” is sufficient to express “converges to.” Looking at the proof more critically, it could be found the argument appears just as a collection of statements that are not connected to each other.

Another argument for proving that $\lim_{x \rightarrow 0} \sin \frac{1}{x}$ does not exist is given by another student, Mary.

With a different approach, she concluded that $\lim_{x \rightarrow 0} \sin \frac{1}{x} = 0$ (► Figure 2). Representations and image are problematic in this argument. The other problem concerns the grammar structure, phrases, and vocabulary. This proof uses the squeeze theorem [57], however, it is still, to a certain extent, influenced

by the theorem of divergence criteria [57]. At the beginning of the argument, she expressed that the boundary functions involved in the inequality $x_n \leq y_n \leq z_n$ must not equal to 0, although she mistakenly wrote $y_n \neq 0$ instead of $z_n \neq 0$. It is clearly seen that Mary faces the constraint of semiotic system. Regarding the symbolic representation, she wrote how the symbols read: "... lim kiri dan kanan $\sin(1/x)$ untuk $x \rightarrow 0$ nilainya sama yaitu 0 ..." [left- and right-hand limits of $\sin(1/x)$ for $x \rightarrow 0$ have the same value namely 0], instead of the written expression itself: $\lim_{x \rightarrow 0^-} \sin \frac{1}{x} = 0 = \lim_{x \rightarrow 0^+} \sin \frac{1}{x}$. This result is not correct, actually. Another expression is " $\lim x \rightarrow 0^+$ maka $-x = \lim x \rightarrow 0^+$ sehingga $x = 0$ " [$\lim x \rightarrow 0^+$ then $-x = \lim x \rightarrow 0^+$ so that $x = 0$] which does not follow the conventional grammatical structure. Regarding the graph illustrating the squeeze theorem, it is inaccurately depicting the function $y = f(x) = \sin \frac{1}{x}$. This visual image could be considered as the sketch of $y = f(x) = x \sin \frac{1}{x}$ which is bounded by $y = -|x|$ and $y = |x|$. So, in this particular student proof, the grammar structure, technical vocabularies and phrases as well the symbolic representation and visual image become the constraints that make the argument invalid.

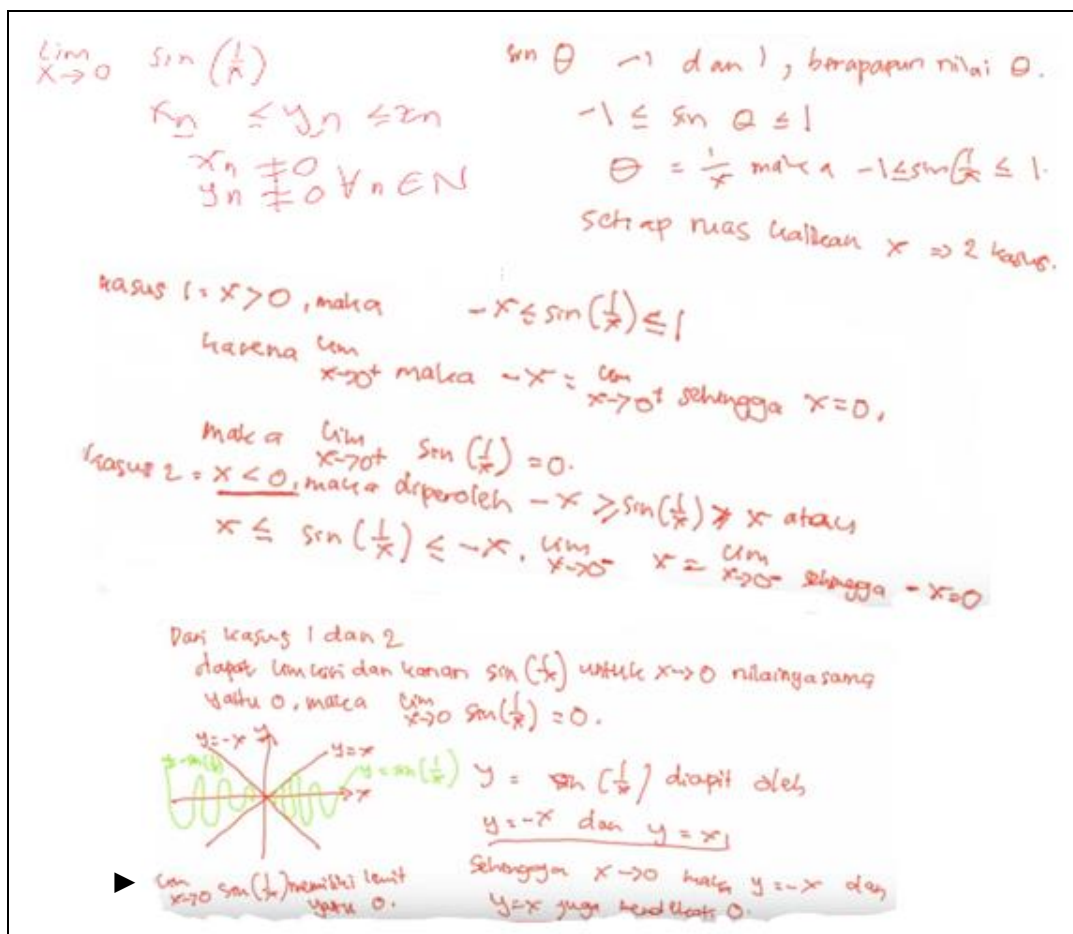


Figure 2. Mary's proof

Diketahui : $\lim_{x \rightarrow 0} \sin \frac{1}{x}$
 Akan dibuktikan : $\lim_{x \rightarrow 0} \sin \frac{1}{x}$ tidak ada
 Akan ditunjukkan: Ada barisan $(x_n) \rightarrow 0$ dan $(y_n) \rightarrow 0$
 dengan $x_n \neq 0$ dan $y_n \neq 0$
 Ada barisan $(x_n) \rightarrow 0$ dan $(y_n) \rightarrow 0$ dengan $x_n \neq 0$ dan $y_n \neq 0, \forall n \in \mathbb{N}$.
 Tetapi $\lim_{x \rightarrow 0} \sin \frac{1}{x_n} \neq \lim_{x \rightarrow 0} \sin \frac{1}{y_n}$
 $\lim_{x \rightarrow 0} \sin \frac{1}{x_n} \rightarrow 1 \neq \lim_{x \rightarrow 0} \sin \frac{1}{y_n} \rightarrow 0$
 Pilih $x_n = \frac{2}{(4n-3)\pi}$ maka $x_n \rightarrow 0$ Pilih $y_n = \frac{1}{n\pi}$ maka $y_n \rightarrow 0$
 $\Rightarrow \lim_{x \rightarrow 0} \sin \left(\frac{1}{x_n} \right)$ $\Rightarrow \lim_{x \rightarrow 0} \sin \left(\frac{1}{y_n} \right)$
 $\Rightarrow \lim_{n \rightarrow \infty} \sin \left(\frac{1}{\frac{2}{(4n-3)\pi}} \right)$ $\Rightarrow \lim_{n \rightarrow \infty} \sin \left(\frac{1}{\frac{1}{n\pi}} \right)$
 $\Rightarrow \lim_{n \rightarrow \infty} \sin \left(\frac{(4n-3)\pi}{2} \right)$ $\Rightarrow \lim_{n \rightarrow \infty} \sin(n\pi)$
 $\Rightarrow 1$ Konvergen ke 1 $\Rightarrow 0$ Konvergen ke 0
 Sehingga kita memperoleh bahwa $(x_n) \rightarrow 0$ dan $(y_n) \rightarrow 0$ dengan
 $x_n \neq 0$ dan $y_n \neq 0$, tetapi $\lim_{x \rightarrow 0} \sin \left(\frac{1}{x_n} \right) \neq \lim_{x \rightarrow 0} \sin \left(\frac{1}{y_n} \right)$
 Jadi kesimpulannya, $\lim_{x \rightarrow 0} \sin \left(\frac{1}{x} \right)$ tidak ada

Figure 3. Lina's proof

The proof constructed by Lina appears very neatly written (Figure 3). However, there are some problems concerning the use of logical connectives, assumptions, and symbolic representations. In terms of the representations, the symbolic expression $\lim_{x \rightarrow 0} \sin \frac{1}{x} \rightarrow 1 \neq \lim_{x \rightarrow 0} \sin \frac{1}{y} \rightarrow 0$ has conceptual and representational errors. Representationally, if L is the limit of a function, then it must use '=' instead of ' $\rightarrow$ '. Conceptually, it is not this ordinary limit to be investigated, but $\lim \left(\sin \frac{1}{x_n} \right) \neq \lim \left(\sin \frac{1}{y_n} \right)$, which means that the two sequences, $\left(\sin \frac{1}{x_n} \right)$ and $\left(\sin \frac{1}{y_n} \right)$ are convergent to different real numbers. Lina seems to be unaware of the hidden assumption in the symbolic expression of convergent sequence, namely, it is not an ordinary limit, but a limit at infinity written without " $n \rightarrow \infty$ " attached to "lim". As could be found in this proof, $\lim_{n \rightarrow \infty} \sin \left(\frac{(4n-3)\pi}{2} \right) = 1$, but it does not imply that $\left(\sin \left(\frac{(4n-3)\pi}{2} \right) \right) \rightarrow 1$. The other problem is the use of logical connectives. In comparison to the proof of Andy, this proof tries to make the connection among statements within the argument clear. However, the symbol of conditional is used inappropriately.

The three arguments presented above are intended to prove that $\lim_{x \rightarrow 0} \sin \frac{1}{x}$ does not exist. Using the divergent criteria theorem [57], the proof could be constructed by determining two sequences, say, $(x_n) \neq 0$ and $(y_n) \neq 0$, that are convergent to 0, $(x_n) \rightarrow 0$ and $(y_n) \rightarrow 0$, but the sequence $\left(\sin \frac{1}{x_n} \right) \rightarrow K$ and $\left(\sin \frac{1}{y_n} \right) \rightarrow L$, and $K \neq L$. This theorem is the negation of the sequence criteria

[57] for the existence of limit simply stating that: for function $f: A \rightarrow \mathbf{R}$ and c is a cluster point of A , $\lim_{x \rightarrow c} f = L$ if and only if for every sequence (x_n) in A that converges to c such that $x_n \neq c$ for all natural numbers n , the sequence $(f(x_n))$ converges to L . The students attempt to prove the non-existence of $\lim_{x \rightarrow 0} \sin \frac{1}{x}$ by formulating two sequences $(x_n) \neq 0$ and $(y_n) \neq 0$. All them succeed to obtain

the two sequences satisfying the requirement that both sequences are convergent to 0. The problems experienced by the students are evident in further steps towards to conclusion of the proofs.

An argument is an essay containing paragraphs that describe the logical flow of deductive-axiomatic reasoning to show the truth of a mathematical statement. The proof argument is a special genre characterized by its formality with grammatical structure aimed at communicating mathematical reasoning and thinking [43]. An argument for proof must conform to the conventions of mathematical writing where semiotic systems, representations, terminologies, grammatical structures, principles, assumptions, and logic are interrelated complexly. Therefore, the language aspects of mathematics become constraints for the students in their attempt to construct a complete argument for proof.

Representations in mathematics learning, particularly in mathematics communication, are central for they are the only visible expressions of the students' thinking and conceptions [58]. Mathematical representations are an integral part of mathematics language or register [45] used to express mathematical ideas in which vocabularies, terms, phrases are employed with dense, different mathematical meanings. The representation use is one challenge in constructing an argument for proof. In addition, the quality of brevity [47, 48] and clarity [26] avoiding redundant and ineffective expression intensify the linguistic constraint in proving activity. The mathematical operations or principles which are left implicit [59] also contribute to the students' difficulties. As exemplified in a student's constructed proof (Figure 3), the implicit rule in the sequence convergence as a limit at infinity confused her so she conclude that $\lim_{x \rightarrow 0} f(x_n) = L$ is equivalent to $(f(x_n)) \rightarrow L$.

Mathematical logic as the foundation for mathematical thinking and reasoning presents another challenge for the students. Uncovering the meaning of mathematical statements and translating mathematical statements into symbolic logic expressions [60] are challenging for some students. They could not follow the interpretation and meanings prevailing in logic [58]. Figure 3 shows how the student might misunderstand the meaning of ' $\Rightarrow$, conditional' and then use it carelessly. It is urgent for the students to understand the meaning of mathematical symbols before engaging themselves in the mathematical study [61].

The prior knowledge the students bring into their mathematics classes is considered as a good capital for their learning. However, the capitalization of the students' prior knowledge should be done carefully. Impressive prior experience influences the students' learning [62]. Further, it is argued that an individual's mental capacity shaped by their previous experiences could be either advantageous or disadvantageous to mathematical thinking [62]. This disadvantageous effect could be observed on the proof constructed by Mary (Figure 2) where she used the squeeze theorem incorrectly to prove the given limit.

5. Conclusions

Proof and proving are crucial in mathematical study. Constructing valid arguments for proof is difficult as it is affected by various factors, one of which is the language constraints. More preparation is always needed by the students before studying mathematical proofs. Students need to develop a conception of the genre and the agreed rules of mathematical proof in order to enable them to engage meaningfully in the proving activities.

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