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SIR Model Analysis for Transmission of Dengue Fever Disease with Climate Factors Using Lyapunov Function

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Abstract: The aim of this paper is to discuss the dynamic system of SIR model with climate factors for transmission characteristics of dengue fever in closed population. Human population is divided into three types, which are susceptible population, infected population, and recovered population, while mosquito population is divided into two kinds that are susceptible population and infected population contaminated by dengue fever virus. Five nonlinear differential equations are analyzed to obtain the equilibrium of the system and the basic reproduction number R_0. Next, the stability of the equilibrium of the system is dissected with three theorems using lyapunov. The results will be in the form of the disease-free equilibrium, the endemic equilibrium, and the basic reproduction number. The basic reproduction number R_0 is influenced by climate factors. If $R_0 \leq 1$, the disease-free equilibrium will be asymptotically stable, whereas if $R_0 > 1$, the endemic equilibrium will become asymptotically stable instead. Based on the results, the dengue fever transmission is affected greatly by the climate of a region.

1. Introduction

Infectious diseases were estimated to vanish along with the development of medical research for vaccines, antibiotics, and life quality enhancement in the middle of 20th century. However, in fact, in the beginning of 21st century, infectious diseases still caused sorrow in the developing countries. Malaria, jaundice, AIDS, and Ebola are some of the culprits. Among all of those diseases, one of the easily transferred diseases is dengue fever. Dengue fever spread in the Southeast Asia and all over the world, particularly in the tropical climate countries [1,2]

Some articles have discussed the dengue fever transmission model, SIR model, and looked for the disease-free equilibrium and the three endemic equilibriums to reduce the sufferers of dengue fever [4], modelling dengue fever by doing detection and prediction of dengue fever in Semarang [3], simulating dengue fever transmission in Selangor (Malaysia) [5], modelling dengue fever transmission by paying attention on the parameter of the mobility of citizens between two cities [7,8,9]. In this article, the mathematics modelling for dengue fever transmission by including climate factor as an influential parameter toward model, disease-free equilibrium and endemic equilibrium analyzed by using Lyapunov function following steps [6,13] that discovered the global equilibrium for SIR and SEIR mathematics model in dengue fever transmission.

2. Formulation Of Sir Model With Climate Factors

Human population is divided into three classes, Susceptible, Infected, and Recovery (Immune), while mosquito population is divided into two classes - Susceptible and Infected. At time t, we denote the susceptible humans for dengue fever, the infectious humans, the recovered humans in the particular range of time, susceptible mosquitos for dengue fever, and the infectious mosquitos as S_H, I_H, R_H, S_V, and I_V respectively. No immune class exists for mosquitos because the infected period ends when the mosquitos die. $N_H = S_H + I_H + R_H$ and $N_V = S_V + I_V = \dfrac{A}{\mu_V}$ are the sum of human population and vector at time t. μ_H is birth and death rate for humans and μ_V is for mosquitos.

The alteration in each groups of humans and mosquitos with climate factor α can be interpreted into Figure 1 as follows:
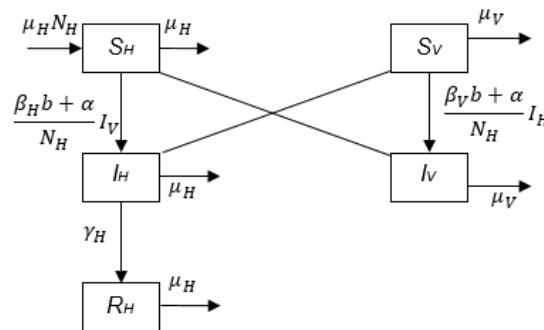

Figure 1. Scheme of human population and vector for SIR model.

Scheme of human population and vector in Figure 1 can be shown into mathematics models that form nonlinear differential equations as seen in equation (1). From those models, the disease-free equilibrium and the endemic equilibrium will be obtained by using Lyapunov function.

$$
\begin{cases}
S_H' = \mu_H N_H - \left(\dfrac{\beta_H b + \alpha}{N_H}\right) S_H I_V - \mu_H S_H \\[2ex]
I_H' = \left(\dfrac{\beta_H b + \alpha}{N_H}\right) S_H I_V - (\gamma_H + \mu_H) I_H \\[2ex]
\quad R_H' = \gamma_H I_H - \mu_H R_H \\[2ex]
S_V' = \mu_V N_V - \left(\dfrac{\beta_V b + \alpha}{N_H}\right) S_V I_H - \mu_V S_V \\[2ex]
\quad I_V' = \left(\dfrac{\beta_V b + \alpha}{N_H}\right) S_V I_H - \mu_V I_V
\end{cases}
\tag{1}
$$

Since $N_V = S_V + I_V = \dfrac{A}{\mu_V}$, equation (1) can be simplified as that in equation (2)

$$\begin{cases} S_H' = \mu_H N_H - \left(\dfrac{\beta_H b + \alpha}{N_H}\right) S_H I_V - \mu_H S_H \\[2ex] I_H' = \left(\dfrac{\beta_H b + \alpha}{N_H}\right) S_H I_V - (\gamma_H + \mu_H) I_H \\[2ex] \qquad R_H' = \gamma_H I_H - \mu_H R_H \\[2ex] \quad S_V' = A - \left(\dfrac{\beta_V b + \alpha}{N_H}\right) S_V I_H - \mu_V S_V \\[2ex] \quad I_V' = \left(\dfrac{\beta_V b + \alpha}{N_H}\right) S_V I_H - \mu_V I_V \end{cases} \qquad (2)$$

All of the parameters and the variables of the models are non-negative. They are clearly shown in the system (2), the non-negatif octant R_+^5 is positively invariant. Related to the system (2), we can obtain the following theorems.

Theorem 1. Let $(S_H(t) > 0, I_H(t) > 0, R_H(t) > 0, S_V(t) > 0, I_V(t) > 0)$ be the solution of the system (2) with the initial condition $(S_{0H}, I_{0H}, R_{0H}, S_{0V}, I_{0V})$ and set D

$$D = \left\{ (S_H, I_H(t), R_H(t), S_V, I_V) \in R_+^5, F_1 \le N_H, F_2 \le \frac{A}{\mu_V} \right\} \qquad (3)$$

for model (2), D is a positively invariant set covering all of the solutions in R_+^5.

Proof. Consider the function below be the future Lyapunov function:

$$F(t) = (F_1(t), F_2(t)) = (S_H + I_H + R_H, S_V + I_V)$$

Next, $F(t)$ will be derived according to time (t)

$$F'(t) = (F_1'(t), F_2'(t)) = (S_H' + I_H' + R_H', S_V' + I_V')$$

$$= \begin{pmatrix} \mu_H N_H - \left(\frac{\beta_H b + \alpha}{N_H}\right) S_H I_V - \mu_H S_H + \left(\frac{\beta_H b + \alpha}{N_H}\right) S_H I_V - (\gamma_H + \mu_H) I_H + \gamma_H I_H - \mu_H R_H, \\ A - \left(\frac{\beta_V b + \alpha}{N_H}\right) S_V I_H - \mu_V S_V + \left(\frac{\beta_V b + \alpha}{N_H}\right) S_V I_H - \mu_V I_V \end{pmatrix}$$

$$= (\mu_H N_H - \mu_H (S_H + I_H + R_H), A - \mu_V (S_V + I_V))$$

$$= (\mu_H N_H - \mu_H F_1, A - \mu_V F_2)$$

Therefore, we get

$$\begin{cases} F_1'(t) = \mu_H N_H - \mu_H F_1 \le 0 \\ F_2'(t) = A - \mu_V F_{21} \le 0 \end{cases} \qquad (4)$$

From equation (4), $F'(t) \le 0$ indicating D is a positively invariant set. By solving equation (4), we get

$$0 \le (F_1(t), F_2(t)) \le \left(N_H + F_1(0) e^{-\mu_H t}, \frac{A}{\mu_V} + F_2(0) e^{\mu_V t} \right)$$

where $F_1(0)$ and $F_2(0)$ are the values of $F_1(t)$ and $F_2(t)$.

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where $F_1(0)$ and $F_2(0)$ are the values of $F_1(t)$ and $F_2(t)$.

From $t \longrightarrow \infty, 0 \leq (F_1(t), F_2(t)) \leq \left(N_H, \dfrac{A}{\mu_V} \right)$, we can conclude that D is a positively invariant set. This

theorem guarantees that in a place, where in the beginning no virus bearer of dengue fever exists, turns into a contaminated place after the susceptible population, the not infected yet population (S_H), the infected population (I_H), and the recovered population (R_H) were found.

3. Sir Model Stability Analysis

By applying Diekhmann and Heesterbeek method [10, 11], Basic Reproduction can be obtained from system (2)

$$R_0 = \frac{A(b(b\beta_H \beta_V + \beta_H \alpha + \beta_V \alpha) + \alpha^2)}{N_V \mu_V (\mu_H + \gamma_H)} \tag{5}$$

System (2) has two equilibriums: $R_0 \leq 1$ of which the only equilibrium is the disease-free equilibrium $K^* = (S_H^*, I_H^*, R_H^*, S_V^*, I_H^*) = \left(N_H, 0, 0, \dfrac{A}{\mu_V}, 0 \right) \in D$ and $R_0 > 1$ of which the endemic equilibriums can be obtained, $K^{**} = (S_H^{**}, I_H^{**}, R_H^{**}, S_V^{**}, I_H^{**}) \in D$ where $S_H^{**}, I_H^{**}, R_H^{**}, S_V^{**}, I_H^{**} > 0$ with

$$S_H^{**} = \frac{\mu_V N_H^2 (b\beta_V \mu_H + \mu_H \mu_V + \mu_H \alpha + \mu_V \gamma_H)}{(b\beta_V + a)(Ab\beta_H + N_H \mu_H \mu_V + A\alpha)}$$

$$I_H^{**} = \frac{N_H \mu_H \left(Ab^2 \beta_V \mu_H + Ab\beta_H \alpha + Ab\beta_V \alpha - N_H \mu_H \mu_V^2 + A\alpha^2 \right)}{(b\beta_V + a)\left(Ab\beta_H \mu_H + Ab\beta_H \gamma_H + N_H \mu_H^2 \mu_V + N_H \mu_H \mu_V \gamma_H + A\mu_H \alpha + A\alpha\gamma_H \right)}$$

$$R_H^{**} = \frac{N_H \left(Ab^2 \beta_V \beta_H + Ab\beta_H \alpha + Ab\beta_V \alpha - - N_H \mu_H \mu_V^2 + A\alpha^2 \right)\gamma_H}{(b\beta_V + a)\left(Ab\beta_H \mu_H + Ab\beta_H \gamma_H + N_H \mu_H^2 \mu_V + N_H \mu_H \mu_V \gamma_H + A\mu_H \alpha + A\alpha\gamma_H \right)} \tag{6}$$

$$S_V^{**} = \frac{Ab\beta_H \mu_H + Ab\beta_H \gamma_H + N_H \mu_H^2 \mu_V + N_H \mu_H \mu_V \gamma_H + A\mu_H \alpha + A\alpha\gamma_H}{b^2 \beta_V \beta_H \mu_H + b\beta_H \mu_H \mu_V + b\beta_H \mu_H \alpha + b\beta_H \mu_H \gamma_H + b\beta_V \mu_H \alpha + \mu_H \mu_V \alpha + \mu_H \alpha^2 + \mu_V \alpha\gamma_H}$$

$$I_V^{**} = \frac{\mu_H \left(Ab^2 \beta_V \mu_H + Ab\beta_H \alpha + Ab\beta_V \alpha - N_H \mu_H \mu_V^2 + A\alpha^2 \right)}{\left(Ab\beta_H \mu_H + Ab\beta_H \gamma_H + N_H \mu_H^2 \mu_V + N_H \mu_H \mu_V \gamma_H + A\mu_H \alpha + A\alpha\gamma_H \right)\mu_V}$$

System (2) always has a disease-free equilibrium $K^* = (S_H^*, I_H^*, R_H^*, S_V^*, I_H^*) = \left(N_H, 0, 0, \dfrac{A}{\mu_V}, 0 \right) \in D$

that shows the no dengue fever situation.

Teorema 2. If $R_0 \leq 1$, the disease-free equilibrium $K^* = (S_H^*, I_H^*, R_H^*, S_V^*, I_H^*) = \left(N_H, 0, 0, \dfrac{A}{\mu_V}, 0 \right) \in D$ is

asymptotically stable in D with assumptions

$$\begin{cases} \mu_H = \left(\dfrac{b\beta_V + \alpha}{N_H} \right) S_V^* \\ \mu_V = \left(\dfrac{b\beta_H + \alpha}{N_H} \right) S_V^* \end{cases} \tag{7}$$

Proof. Consider the function below be the candidate of Lyapunov function

$$G(t) = (S_H - S_H^* \ln S_H) + I_H + R_H + \left(S_V - S_V^* \ln S_V \right) + I_V \tag{8}$$

Function (8) is derived to time (t), so

$$G'(t) = S_H'(1 - \frac{S_H^*}{S_H}) + I_H' + R_H' + S_V'\left(1 - \frac{S_V^*}{S_V}\right) + I_V'$$

$$= \left(\left(\mu_H N_H - \left(\frac{\beta_H b + \alpha}{N_H}\right)S_H I_V - \mu_H S_H\right)\left(1 - \frac{S_H^*}{S_H}\right)\right) + \left(\left(\frac{\beta_H b + \alpha}{N_H}\right)S_H I_V - (\gamma_H + \mu_H)I_H\right) \quad = \mu_H N_H\left(1 - \frac{S_H^*}{S_H}\right) + I_V\left(\left(\frac{\beta_H b + \alpha}{N_H}\right)S_V^* - \mu_V\right) + I_H\left(\left(\frac{\beta_V b + \alpha}{N_H}\right)S_V^* - \mu_H\right) + \mu_H S_H^*\left(1 - \frac{S_H}{S_H^*}\right) \quad (9)$$

$$+ (\gamma_H I_H - \mu_H R_H) + \left(\left(A - \left(\frac{\beta_V b + \alpha}{N_H}\right)S_V I_H - \mu_V S_V\right)\left(1 - \frac{S_V^*}{S_V}\right)\right) + \left(\left(\frac{\beta_V b + \alpha}{N_H}\right)S_V I_H - \mu_V I_V\right) \quad + A\left(1 - \frac{S_V^*}{S_V}\right) + + \mu_V S_V^*\left(1 - \frac{S_V}{S_V^*}\right) - \mu_H R_H$$

By using assumption in (8), we get

$$G'(t) = \mu_H N_H\left(1 - \frac{S_H^*}{S_H}\right) + + \mu_H S_H^*\left(1 - \frac{S_H}{S_H^*}\right) + A\left(1 - \frac{S_V^*}{S_V}\right) + + \mu_V S_V^*\left(1 - \frac{S_V}{S_V^*}\right) - \mu_H R_H \tag{10}$$

Based on the disease-free equilibrium, we have $S_H^* = N_H$ and $S_V^* = \dfrac{A}{\mu_V}$, then equation (10) can be written as

$$G'(t) = \mu_H N_H\left(1 - \frac{S_H^*}{S_H}\right) + + \mu_H N_H\left(1 - \frac{S_H}{S_H^*}\right) + A\left(1 - \frac{S_V^*}{S_V}\right) + + A\left(1 - \frac{S_V}{S_V^*}\right) - \mu_H R_H$$

$$= \mu_H N_H\left(2 - \frac{S_H^*}{S_H} - \frac{S_H}{S_H^*}\right) + A\left(2 - \frac{S_V^*}{S_V} - \frac{S_V}{S_V^*}\right) - \mu_H R_H$$

$$G'(t) = -\mu_H N_H\frac{\left(S_H - S_H^*\right)^2}{S_H S_H^*} - A\frac{\left(S_V - S_V^*\right)^2}{S_V S_V^*} - \mu_H R_H \tag{11}$$

The result is $G'(t) \le 0$. By using Lasalle's Extension to Lyapunov's method [13], the compact set for each solution is in the largest invariant set with $S_H = S_H^*$, $S_V = S_V^*$ and $R_H = 0$ is singleton $\{P^*\}$. However, it can be concluded that the disease-free equilibrium, P^*, is asymptotically stable in D which becomes the proof of the theorem.

4. The Stability of Endemic Equilibrium

System (2) owns some endemic equilibriums K^{**} that indicate the existence of dengue fever sufferer.

Theorem 3. If $R_0 \le 1$, the endemic equilibrium K^{**} is asymptotically stable in D with assumptions as follow

$$\begin{cases} N_H = S_H^{**} \\ \mu_H = \dfrac{A}{S_V^{**}} \\ b = \dfrac{A(\gamma_H + \mu_H) - \alpha p\left(S_V^{**}\right)^2}{p\beta_V\left(S_V^{**}\right)^2} \end{cases} \tag{12}$$

with

$$\begin{cases} p = \dfrac{\beta_H b + \alpha}{N_H} \\ q = \dfrac{\beta_V b + \alpha}{N_H} \end{cases} \tag{13}$$

Proof. Consider the function below be the candidate of Lyapunov function

$$W(t) = (S_H - S_H^* \ln S_H) + I_H + \left(\frac{\gamma_H + \mu_H}{qS_V^{**}}\right)\left(S_V - S_V^* \ln S_V\right) + \left(\frac{\gamma_H + \mu_H}{qS_V^{**}}\right)I_V \tag{14}$$

Function (14) is derived according to time (t), so we can get

$$W'(t) = S_H'\left(1 - \frac{S_H^*}{S_H}\right) + I_H' + \left(\frac{\gamma_H + \mu_H}{qS_V^{**}}\right)S_V'\left(1 - \frac{S_V^*}{S_V}\right) + \left(\frac{\gamma_H + \mu_H}{qS_V^{**}}\right)I_V'$$

$$= \left(\left(\mu_H N_H - \left(\frac{\beta_H b + \alpha}{N_H}\right)S_H I_V - \mu_H S_H\right)\left(1 - \frac{S_H^*}{S_H}\right)\right) + \left(\left(\frac{\beta_H b + \alpha}{N_H}\right)S_H I_V - (\gamma_H + \mu_H)I_H\right)$$

$$+ \left(\frac{\gamma_H + \mu_H}{qS_V^{**}}\right)\left(\left(A - \left(\frac{\beta_V b + \alpha}{N_H}\right)S_V I_H - \mu_V S_V\right)\left(1 - \frac{S_V^*}{S_V}\right)\right)$$

$$+ \left(\frac{\gamma_H + \mu_H}{qS_V^{**}}\right)\left(\left(\frac{\beta_V b + \alpha}{N_H}\right)S_V I_H - \mu_V I_V\right)$$

$$= \left(\mu_H N_H \left(1 - \frac{S_H^*}{S_H}\right) + \mu_H S_H^* \left(1 - \frac{S_H}{S_H^*}\right) \right) + I_V \left(\left(\frac{\beta_H b + \alpha}{N_H} \right) S_V^* - \left(\frac{\gamma_H + \mu_H}{q S_V^{**}} \right) \mu_V \right)$$

$$+ I_H \left(\left(\frac{\gamma_H + \mu_H}{q S_V^{**}} \right) \left(\frac{\beta_V b + \alpha}{N_H} \right) S_V^* - \mu_H \right) + \left(\frac{\gamma_H + \mu_H}{q S_V^{**}} \right) A \left(\left(1 - \frac{S_V^*}{S_V}\right) + \left(1 - \frac{S_V}{S_V^*}\right) \right) \tag{15}$$

By using assumption 12, equation (15) becomes

$$W'(t) = \left(\mu_H N_H \left(1 - \frac{S_H^*}{S_H}\right) + \mu_H N_H \left(1 - \frac{S_H}{S_H^*}\right) \right) + \mu_V \left(\frac{\gamma_H + \mu_H}{q} \right) \left(\left(1 - \frac{S_V^*}{S_V}\right) + \left(1 - \frac{S_V}{S_V^*}\right) \right)$$

$$= \left(\mu_H N_H \left(2 - \frac{S_H^*}{S_H} - \frac{S_H}{S_H^*}\right) \right) + \mu_V \left(\frac{\gamma_H + \mu_H}{q} \right) \left(2 - \frac{S_V^*}{S_V} - \frac{S_V}{S_V^*}\right)$$

$$W'(t) = -\mu_H N_H \left(\frac{(S_H - S_H^*)^2}{S_H S_H^*} \right) - \mu_V \left(\frac{\gamma_H + \mu_H}{q} \right) \left(\frac{(S_V - S_V^*)^2}{S_V S_V^*} \right) \tag{16}$$

Condition (16) guarantees $W'(t) \leq 0$ for all $(S_H, I_H, R_H, S_V, I_V) \in D$ and $W'(t) = 0$ if and only if $S_H = S_H^{**}, I_H = I_H^{**}, R_H = R_H^{**}, S_V = S_V^{**}$ and $I_V = I_V^{**}$. It makes point P^{**} becoming the only positively invariant set in system (2) that fully exists in $L = \{(S_H, I_H, R_H, S_V, I_V), S_H = S_H^{**}, I_H = I_H^{**}, R_H = R_H^{**}, S_V = S_V^{**}, I_V = I_V^{**}\}$ and based on the asymptotically stable theorem [12], the endemic equilibrium P^{**} satisfies the theorem in *D*. From this fact, Theorem 3 has proven.

5. Conclusion

There are three theorems obtained in this article about the existence of dengue fever transmission, the disease-free status, and the endemic status of dengue fever transmission. The stability analysis of the disease-free equilibrium and the endemic equilibrium is undertaken by using lyapunov function. If $R_0 \leq 1$, the disease-free equilibrium is asymptotically stable, then the disease disappears from the population means that no one suffers from dengue fever. Meanwhile, if $R_0 > 1$, the endemic equilibrium becomes asymptotically stable, which means the infected human will contaminate the disease to at least one individual in the population.

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